

BILIPSCHITZ EMBEDDING OF HOMOGENEOUS FRACTALS

FAN LÜ, MAN-LI LOU, ZHI-YING WEN, AND LI-FENG XI

ABSTRACT. In this paper, we introduce a class of fractals named homogeneous sets based on some measure versions of homogeneity, uniform perfectness and doubling. This fractal class includes all Ahlfors–David regular sets, but most of them are irregular in the sense that they may have different Hausdorff dimensions and packing dimensions. Using Moran sets as main tool, we study the dimensions, bilipschitz embedding and quasi-Lipschitz equivalence of homogeneous fractals.

1. INTRODUCTION

It is well known that self-similar sets and self-conformal sets satisfying the open set condition (**OSC**) are always *Ahlfors–David regular* [9, 13]. We say that a compact subset A of metric space (X, d) is Ahlfors–David s -regular with $s \in (0, \infty)$, if there is a Borel measure μ supported on A and a constant $c \geq 1$ such that for all $x \in A$ and $0 < r \leq |A|$,

$$c^{-1}r^s \leq \mu(B(x, r)) \leq cr^s, \quad (1.1)$$

where $B(x, r)$ is the closed ball centered at x with radius r and $|\cdot|$ denotes the diameter of a set. For an Ahlfors–David s -regular set A , $0 < \mathcal{H}^s(A) < \infty$ and $\dim_H A = \dim_P A = s$, i.e., its Hausdorff dimension and packing dimension are the same.

Ahlfors–David regularity is a weak notion of *homogeneity* [4]. We give another *measure version* of homogeneity, i.e., there is a constant $\lambda \geq 1$ such that for all $x_1, x_2 \in A$ and $0 < r \leq |A|$,

$$\lambda^{-1} \leq \frac{\mu(B(x_1, r))}{\mu(B(x_2, r))} \leq \lambda. \quad (1.2)$$

Naturally, (1.2) holds for all Ahlfors–David regular sets.

We also need two other notions, *uniform perfectness* and *doubling*, which play important roles in the research of metric spaces. For example, Proposition 15.11 of [4] shows that if a compact metric space is uniformly perfect, doubling and uniformly disconnected, then it is quasisymmetrically equivalent to a symbolic system Σ_2 .

We notice that any Ahlfors–David regular set A is *uniformly perfect* (see, e.g., [1] and [12]), i.e., it contains more than one point and there exists a constant $t \in (0, 1)$ such that $[B(x, r) \setminus B(x, tr)] \cap A \neq \emptyset$ for all $x \in A$, $0 < r \leq |A|$. For the *measure*

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version of uniform perfectness, we obtain an alternative condition: there exists a constant $\kappa_1 < 1$ such that

$$\inf_{x \in A, r \leq |A|} \frac{\mu(B(x, r))}{\mu(B(x, \kappa_1 r))} > 1. \quad (1.3)$$

It follows from (1.1) that any Ahlfors–David regular set satisfies (1.3).

In a metric space, the notion of *doubling* describes that any closed ball of radius r can be covered by no more than M balls of radius $r/2$, where M is a constant. The notion of doubling also has *measure version*, see e.g. [11] and [21]. For compact subsets in metric space, these two versions are equivalent. It follows from (1.1) that any Ahlfors–David regular measure is doubling, i.e., there exists a constant $T \geq 1$ such that $\mu(B(x, r)) \leq T\mu(B(x, r/2))$ for all $x \in A$, $0 < r \leq |A|$, i.e., for $\kappa_2 = 1/2$,

$$\sup_{x \in A, r \leq |A|} \frac{\mu(B(x, r))}{\mu(B(x, \kappa_2 r))} < \infty. \quad (1.4)$$

Simulating the homogeneity, uniform perfectness and doubling by (1.2), (1.3) and (1.4), we can define a large class of fractals, which are not so good as Ahlfors–David regular sets but *homogeneous* in certain sense.

Definition 1. A compact subset A of metric space (X, d) is said to be **homogeneous**, if $|A| > 0$ and there is a Borel probability measure μ supported on A satisfying:

- (1) There is a constant $\lambda_A \geq 1$, such that for all $x_1, x_2 \in A$ and $0 < r \leq |A|$,

$$\lambda_A^{-1} \leq \frac{\mu(B(x_1, r))}{\mu(B(x_2, r))} \leq \lambda_A; \quad (1.5)$$

- (2) There are constants $\kappa_A \in (0, 1)$ and $1 < \delta_A \leq \Delta_A < \infty$, such that for all $x \in A$ and $0 < r \leq |A|$,

$$\delta_A \leq \frac{\mu(B(x, r))}{\mu(B(x, \kappa_A r))} \leq \Delta_A. \quad (1.6)$$

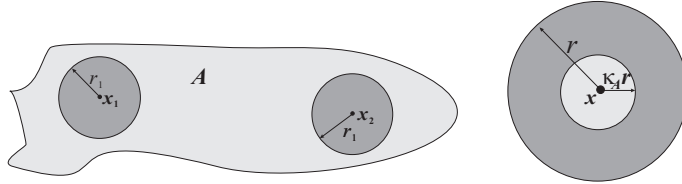


FIGURE 1. Compare the measures of different balls

Remark 1. All Ahlfors–David regular sets are homogeneous. But homogeneous sets may be not Ahlfors–David regular, see Proposition 5 and Example 3 in Section 3.

Remark 2. Any Moran set is homogeneous, see Proposition 2 in Subsection 1.2.

Remark 3. If there exists a point x in A such that $\delta \leq \mu(B(x, r))/\mu(B(x, \kappa r)) \leq \Delta$ holds for all $0 < r \leq |A|$ with constants $\kappa \in (0, 1)$ and $1 < \delta \leq \Delta < \infty$, then it follows from (1.5) that (1.6) holds for any point in A (with some constants κ_A, δ_A and Δ_A).

Remark 4. By (1.6), there are no atoms in A .

There are some fundamental questions about homogeneous fractals:

- How about the **dimensions** of homogeneous fractals? Can we find a **large class of homogeneous fractals** which are not Ahlfors–David regular?
- How about the **bilipschitz embedding** between homogeneous fractals? Which kind of good fractals can be bilipschitz embedded into them?
- Given two homogeneous fractals, when are they bilipschitz equivalent? An alternative but weaker question is of **quasi-Lipschitz equivalence**.

To answer the above questions, we define a function $\alpha_A(x, r)$ for a homogeneous set A as follows:

$$\alpha_A(x, r) = \log \mu(B(x, r)) / \log r \quad \text{for } x \in A, 0 < r \leq |A|. \quad (1.7)$$

Here $\alpha_A(x, r)$ is similar to the function with respect to pointwise dimension.

For any function $g(r)$ defined on $(0, \delta)$ with $\delta > 0$, we focus on the behavior of the function $g(r)$ when $r \rightarrow 0$. In fact, for any function $h(r)$ with

$$|h(r) - g(r)| = O(|\log r|^{-1}), \quad (1.8)$$

we denote $g \sim h$ and define an equivalence class $[g] = \{h : g \sim h\}$. Then, as $\alpha_A(x_1, r) \sim \alpha_A(x_2, r)$ by (1.5), we use $\alpha_A(r)$ to denote any one function in the equivalence class $[\alpha_A(x, r)]$ with $x \in A$. For example, we can take

$$\alpha_A(r) \equiv s \text{ for an Ahlfors–David } s\text{-regular set } A. \quad (1.9)$$

With the help of the function $\alpha_A(r)$ defined above, we can answer the above questions on dimensions, bilipschitz equivalence and quasi-Lipschitz equivalence.

1.1. Dimensions.

Proposition 1. For a homogeneous set A , we have:

- (1) $0 < \liminf_{r \rightarrow 0} \alpha_A(r) \leq \limsup_{r \rightarrow 0} \alpha_A(r) < \infty$ and

$$\dim_H A = \underline{\dim}_B A = \liminf_{r \rightarrow 0} \alpha_A(r), \quad \dim_P A = \overline{\dim}_B A = \limsup_{r \rightarrow 0} \alpha_A(r),$$

where $\dim_P A$ denotes the radius packing dimension of metric space A defined in [3], which coincides with the usual definition when A is a subset of a Euclidean space.

- (2) Suppose $N(A, r)$ is the smallest number of balls with radius r needed to cover A . Let $f_A(r) = \frac{\log N(A, r)}{-\log r}$. Then

$$f_A(r) \sim \alpha_A(x, r) \text{ for any } x \in A. \quad (1.10)$$

These properties show that for a homogeneous set A ,

- The behavior of $\alpha_A(r)$ when $r \rightarrow 0$ is only determined by $N(A, r)$ as in (1.10), i.e., $\alpha_A(x, r) \sim \frac{\log N(A, r)}{-\log r}$, depending on the geometric structure of A and not depending on the choice of the Borel measure μ ;
- The behavior of $\alpha_A(r)$ when $r \rightarrow 0$ plays a role more important than fractal dimensions. We concern not only the dimension values $\liminf_{r \rightarrow 0} \alpha_A(r)$ and $\limsup_{r \rightarrow 0} \alpha_A(r)$, but also the behavior of $\alpha_A(r)$ when $r \rightarrow 0$.

1.2. Moran sets are homogeneous.

Moran sets were first studied in [15] by Moran. We recall this fractal class.

Fix a compact set $J \subset \mathbb{R}^d$ with its interior non-empty. Fix a ratio sequence $\{c_k\}_{k \geq 1}$ and an integer sequence $\{n_k\}_{k \geq 1}$ satisfying $c_k \in (0, 1)$ and $n_k \geq 2$ for all k . For $D_1, D_2 \subset \mathbb{R}^d$, we say that D_1 is geometrically similar to D_2 with ratio r , if there is a similitude S with ratio r such that $D_1 = S(D_2)$. Let $\Omega_0 = \{\emptyset\}$ with the empty word \emptyset , and let $\Omega_k = \{\text{word } i_1 \cdots i_k : \text{for the } t\text{-th letter, } i_t \in \mathbb{N} \cap [1, n_t] \text{ for all } t\}$ for $k \geq 1$. In this paper, we always assume that

$$c_* = \inf_k c_k > 0. \quad (1.11)$$

Suppose there are $J_1, J_2, \dots, J_{n_1} \subset J_\emptyset = J$ geometrically similar to J with ratio c_1 and their interiors being pairwise disjoint. Inductively, for any $i_1 \cdots i_{k-1} \in \Omega_{k-1}$, suppose there are $J_{i_1 \cdots i_{k-1} 1}, J_{i_1 \cdots i_{k-1} 2}, \dots, J_{i_1 \cdots i_{k-1} n_k} \subset J_{i_1 \cdots i_{k-1}}$ geometrically similar to $J_{i_1 \cdots i_{k-1}}$ with ratio c_k and their interiors being pairwise disjoint. Then

$$E = \bigcap_{k=0}^{\infty} \bigcup_{i_1 \cdots i_k \in \Omega_k} J_{i_1 \cdots i_k} \quad (1.12)$$

is called a Moran set. We denote $E \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$.

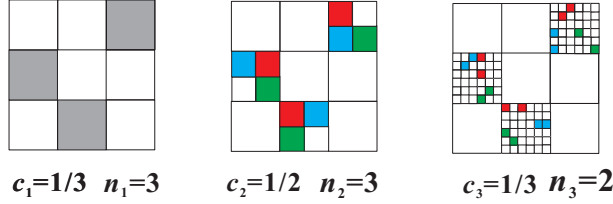


FIGURE 2. The first three steps of the construction of a Moran set with $J = [0, 1]^2$

Many classical self-similar sets are Moran sets. For the Cantor ternary set and the von Koch curve, setting $n_k \equiv 2$ in both cases, letting $c_k \equiv 1/3$ or $c_k \equiv 1/\sqrt{3}$, respectively, and taking J as $[0, 1]$ or a suitable solid triangle, respectively, we get their structures. For details of a more general structure, please refer to [20].

Under the assumption (1.11), we have

Proposition 2. *Any Moran set is homogeneous. Suppose E is a Moran set defined above. Then we can take $\alpha_E(r) = \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}$ if $(c_1 \cdots c_k)|J| < r \leq (c_1 \cdots c_{k-1})|J|$.*

Note that bilipschitz image of a homogeneous set is homogeneous (Lemma 4).

Corollary 1. *Any bilipschitz image of a Moran set is homogeneous.*

1.3. Approximation theorem.

How to describe the distance between two homogeneous sets? As usual, we can use Hausdorff distance d_H . For homogeneous sets A and B in a metric space (X, d) ,

$$d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

where $d(x, B) = \inf_{z \in B} d(x, z)$.

We give a new pseudo-distance. Given homogeneous sets A and B , we consider

$$\chi(A, B) := \limsup_{r \rightarrow 0} \left| \log \frac{\alpha_A(r)}{\alpha_B(r)} \right|.$$

It is easy to check that χ is a pseudo-distance on the space of all homogeneous sets, i.e., $\chi(A, B) \geq 0$, $\chi(A, B) = \chi(B, A)$ and $\chi(A, B) + \chi(B, C) \geq \chi(A, C)$. In fact, $\chi(A, B) = 0$ if and only if $\lim_{r \rightarrow 0} \frac{\alpha_A(r)}{\alpha_B(r)} = 1$, i.e., $\lim_{r \rightarrow 0} (\alpha_A(x, r) - \alpha_B(y, r)) = 0$ for all $x \in A, y \in B$.

Proposition 3. *Given a homogeneous set A , for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\chi(A, B) < \delta$ for a homogeneous set B , then*

$$|\dim_P A - \dim_P B|, |\dim_H A - \dim_H B| < \varepsilon.$$

We can approximate Ahlfors–David regular sets by self-similar sets as in [14]. For homogeneous sets, we replace self-similar sets by Moran sets.

Theorem 1. *Suppose A is a homogeneous set. Then for any $\varepsilon > 0$, we can find a Moran set E in a Euclidean space and a bilipschitz map f from E to A such that*

$$d_H(f(E), A) < \varepsilon \quad \text{and} \quad \chi(E, A) = \chi(f(E), A) < \varepsilon.$$

In particular, if A is a homogeneous set in \mathbb{R}^d , for any $\varepsilon > 0$, we can find a Moran set $F \subset A$ such that

$$d_H(F, A) < \varepsilon \text{ and } 0 \leq (\dim_P A - \dim_P F), (\dim_H A - \dim_H F) < \varepsilon.$$

1.4. Embedding theorem.

Definition 2. *For metric spaces (X, d_X) and (Y, d_Y) , we say that X can be bilipschitz embedded into Y , denoted by $X \hookrightarrow Y$, if there exists an injection $f: X \rightarrow Y$ and a constant $L \geq 1$, such that for all $x_1, x_2 \in X$,*

$$d_X(x_1, x_2)/L \leq d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2).$$

Furthermore, if f is a bijection, we say that X and Y are bilipschitz equivalent.

Mattila and Saaranen [14], Llorente and Mattila [10] studied bilipschitz embeddings between subsets of Ahlfors–David regular sets and self-conformal sets respectively. Inspired by [14], Deng, Wen, Xiong and Xi [5] gave the results on self-similar sets.

In fact, Mattila and Saaranen [14] obtained the following interesting result: For an Ahlfors–David s -regular set A and t -regular set B with $s < t$ and any $\varepsilon > 0$, there exists a self-similar set C_ε such that $\dim_H C_\varepsilon \in (s - \varepsilon, s]$ such that $C_\varepsilon \hookrightarrow A$ and $C_\varepsilon \hookrightarrow B$; furthermore, if A is *uniformly disconnected*, then $A \hookrightarrow B$. An interesting fact is that A is uniformly disconnected if $s < 1$.

It is natural to ask how to generalize the above bilipschitz embedding result to homogeneous sets?

The following lemma is given a straightforward proof in Section 5.1.

Lemma 1. *Let A and B be homogeneous with measures μ and ν , $\alpha_A(r) \sim \alpha_A(x^*, r)$ and $\alpha_B(r) \sim \alpha_B(y^*, r)$ for some $x^* \in A$ and $y^* \in B$, and let $A \hookrightarrow B$. Then for any $x \in A, y \in B$ and $r' < r \leq \min(|A|, |B|)$,*

$$\frac{\mu(B(x, r))}{\mu(B(x, r'))} \leq C \frac{\nu(B(y, r))}{\nu(B(y, r'))} \quad (1.13)$$

where C is an independent constant. Moreover, there is non-decreasing function $\varepsilon: (0, \delta) \rightarrow (0, \infty)$ with $\delta \in (0, 1)$ and $\varepsilon(r) \downarrow 0$ as $r \downarrow 0$ such that

$$\sup_{r' < r_0 r < r < r_0} \left| \frac{\alpha_A(r) \log r - \alpha_A(r') \log r'}{\alpha_B(r) \log r - \alpha_B(r') \log r'} \right| \leq 1 + \varepsilon(r_0) \quad (1.14)$$

for every $r_0 < \delta$.

Using the above lemma, we have

Proposition 4. *There is a homogeneous set B and a number $t \in (0, \dim_H B)$ such that any Ahlfors–David regular set A , e.g., any self-similar set satisfying the strong separation condition (**SSC**), can not be bilipschitz embedded into B , whenever $t < \dim_H A < \dim_H B$.*

Remark 5. *Compare this proposition with the result in [5]: Let A and B be self-similar sets with $\dim_H A < \dim_H B$ and if A satisfies **SSC**, then $A \hookrightarrow B$.*

Now, we give the main result on the bilipschitz embedding.

Theorem 2. *Suppose A, B are homogeneous sets and $\alpha_A(r) \sim \alpha_A(x^*, r)$ and $\alpha_B(r) \sim \alpha_B(y^*, r)$ for some $x^* \in A$ and $y^* \in B$. If*

$$\sup_{r' < r_0 r < r < r_0} \left| \frac{\alpha_A(r) \log r - \alpha_A(r') \log r'}{\alpha_B(r) \log r - \alpha_B(r') \log r'} \right| < 1 \quad (1.15)$$

for some $r_0 < 1$, then for any $\varepsilon > 0$, there exists a homogeneous subset $A_\varepsilon \subset A$ such that $d_H(A_\varepsilon, A) < \varepsilon$, $\chi(A_\varepsilon, A) < \varepsilon$ and A_ε can be bilipschitz embedded into B . Further, if A is uniformly disconnected and (1.15) holds, then $A \hookrightarrow B$.

Remark 6. *If A, B are Ahlfors–David regular with $\dim_H A < \dim_H B$, taking $\alpha_A(r) \equiv \dim_H A$ and $\alpha_B(r) \equiv \dim_H B$, we obtain (1.15).*

Remark 7. *Here A_ε is bilipschitz equivalent to a Moran set. Compared with [14], we use Moran sets to replace self-similar sets.*

Here we say that a compact subset A of a metric space is *uniformly disconnected* [4], if there are constants $C > 1$ and $r^* > 0$ so that for any $x \in A$ and $r < r^*$, there exists a set $E \subset A$ satisfying

$$A \cap B(x, r) \subset E \subset B(x, Cr) \text{ and } d(E, A \setminus E) > r. \quad (1.16)$$

Any self-similar set satisfying **SSC** is uniformly disconnected. Sometimes, we can use the uniform disconnectedness to replace **SSC**.

Lemma 2. *Suppose A is a homogeneous set. If*

$$\sup_{r' < r_0 r < r < r_0} \left| \frac{\alpha_A(r) \log r - \alpha_A(r') \log r'}{\log r - \log r'} \right| < 1$$

for some $r_0 < 1$, then A is uniformly disconnected.

Remark 8. *For Ahlfors–David s -regular set A with $s < 1$, we obtain its uniform disconnectedness by taking $\alpha_A(r) \equiv s$. This is a result of [14]. However, using a Moran set one can find a homogeneous set A with $\dim_H A = \dim_P A < 1$ but which is not uniformly disconnected (see Example 4 in Section 5).*

For any $s \in (0, \infty)$, there exists a self-similar set E with $\dim_H E = s$ in Euclidean space satisfying **SSC**. Since E is Ahlfors–David regular and uniformly disconnected, applying Theorem 2 and Lemma 2 to Ahlfors–David regular sets (Remarks 6 and 8), one can get a result of Mattila and Saaranen [14].

1.5. Equivalence theorem.

Classifying fractals under bilipschitz equivalence is an important topic in geometric measure theory.

Bilipschitz mappings preserve the geometric properties, such as fractal dimensions, Ahlfors–David regularity and uniform disconnectedness. Many works have been devoted to the bilipschitz equivalence of fractals, please refer to [7], [4], [22], [16], [24] and [10]. But even for self-similar sets, Falconer and Marsh [7] pointed out that there are two self-similar sets satisfying **SSC** with the same Hausdorff dimension but which are not bilipschitz equivalent.

Corresponding to bilipschitz equivalence, a weaker notion of quasi-Lipschitz equivalence was introduced in [23]. Under quasi-Lipschitz mapping, information of fractals is preserved in some sense, for example, the fractal dimensions, quasi Ahlfors–David regularity, quasi uniform disconnectedness; see e.g. [19] and [23].

Definition 3. *Two compact metric spaces (X, d_X) and (Y, d_Y) are said to be quasi-Lipschitz equivalent, if there is a bijection $f : X \rightarrow Y$ such that for all $x_1, x_2 \in X$,*

$$\frac{\log d_Y(f(x_1), f(x_2))}{\log d_X(x_1, x_2)} \rightarrow 1 \text{ uniformly as } d_X(x_1, x_2) \rightarrow 0.$$

If we turn to quasi-Lipschitz equivalence, we can say more about the equivalence of homogeneous sets.

Theorem 3. *Suppose homogeneous sets A, B are uniformly disconnected. Then they are quasi-Lipschitz equivalent if and only if $\chi(A, B) = 0$, i.e., $\lim_{r \rightarrow 0} \frac{\alpha_A(r)}{\alpha_B(r)} = 1$.*

If A and B are Ahlfors–David s -regular and t -regular respectively, we note that $\chi(A, B) = 0$ if and only if $s = t$. Using Theorem 3, we get the main results of [18]: Suppose that A and B are Ahlfors–David s -regular and t -regular respectively, and that they are uniformly disconnected; then they are quasi Lipschitz equivalent if and only if they have the same Hausdorff dimension, i.e., $s = t$. In particular, the assumption $s, t < 1$ implies their uniform disconnectedness (see [14] or Remark 8). Then we also get the result of [23]: Two self-conformal sets satisfying **SSC** are quasi-Lipschitz equivalent if and only if they have the same Hausdorff dimension. For example, the self-similar sets in Example 1 are quasi-Lipschitz equivalent.

Example 1. *The Cantor ternary set and the self-similar set $E = (rE) \cup (rE + \frac{1}{2} - \frac{r}{2}) \cup (rE + 1 - r)$ with $r = 3^{-\log 3 / \log 2}$ are quasi-Lipschitz equivalent, although they are not bilipschitz equivalent as shown in [7] by Falconer and Marsh.*

1.6. Results on Moran sets.

For a Moran class $\mathcal{A} = \mathcal{M}(J, \{n_k\}_{k \geq 1}, \{c_k\}_{k \geq 1})$, supposing

$$r \in (r_k |J|, r_{k-1} |J|], r' \in (r_{k'} |J|, r_{k'-1} |J|] \text{ with } k \leq k',$$

where $r_k = c_1 \cdots c_k$, we let

$$\Phi_{\mathcal{A}}(r) = n_1 \cdots n_k \text{ and } \Phi_{\mathcal{A}}(r, r') = \Phi_{\mathcal{A}}(r') / \Phi_{\mathcal{A}}(r) = n_{k+1} \cdots n_{k'}.$$

Applying Proposition 2 to Theorems 2–3 and Lemma 2, we have

Corollary 2. *Let $\mathcal{A} = \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$ and $\mathcal{B} = \mathcal{M}(I, \{m_k\}_k, \{d_k\}_k)$. If*

$$\limsup_{k \rightarrow \infty} \frac{\log n_{k+1} \cdots n_{k+k_0}}{-\log c_{k+1} \cdots c_{k+k_0}} < 1$$

for some $k_0 \geq 1$, then any $E \in \mathcal{A}$ is uniformly disconnected. If $E \in \mathcal{A}$ is uniformly disconnected and

$$\sup_{r' < r_0 r < r < r_0} \frac{\log \Phi_{\mathcal{A}}(r, r')}{\log \Phi_{\mathcal{B}}(r, r')} < 1$$

for some $r_0 < 1$, then $E \hookrightarrow F$ for any $F \in \mathcal{B}$. If $E \in \mathcal{A}$ and $F \in \mathcal{B}$ are uniformly disconnected, then E and F are quasi-Lipschitz equivalent if and only if

$$\lim_{r \rightarrow 0} \frac{\log \Phi_{\mathcal{A}}(r)}{\log \Phi_{\mathcal{B}}(r)} = 1.$$

Example 2. Let $J = [0, 1]$, $c_k \equiv d_k \equiv 1/5$ and $n_k, m_k \in \{2, 3\}$ for all k . Denote

$$\begin{aligned} a_{k,k'} &= \#\{i : n_i = 3 \text{ with } k \leq i \leq k'\}, \\ b_{k,k'} &= \#\{i : m_i = 3 \text{ with } k \leq i \leq k'\}. \end{aligned}$$

Let $E \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$, $F \in \mathcal{M}(J, \{m_k\}_k, \{d_k\}_k)$. Then E, F are uniformly disconnected. It follows from Corollary 2 that if there exist constants k_0 and k_1 such that

$$0 \leq a_{k,k+k_0} < b_{k,k+k_0} \text{ for all } k > k_1,$$

then $E \hookrightarrow F$. Let

$$a_k = \#\{i : n_i = 3 \text{ with } i \leq k\}, \quad b_k = \#\{i : m_i = 3 \text{ with } i \leq k\}.$$

Using Corollary 2 again, we obtain that E and F are quasi-Lipschitz equivalent if and only if

$$\lim_{k \rightarrow \infty} \frac{a_k \log 3 + (k - a_k) \log 2}{b_k \log 3 + (k - b_k) \log 2} = 1,$$

which is equivalent to

$$\lim_{k \rightarrow \infty} \frac{a_k + k}{b_k + k} = 1.$$

We would mention that this paper is quite different from the previous works, e.g. [14], [5], [23] and [18]. For the fractals discussed in this paper, their Hausdorff dimensions and packing dimensions need not be the same; they are more complicated than Ahlfors–David regular sets as in [14], [5], [23] and [18]. We notice that the main tool of this paper is Moran set rather than self-similar set satisfying **OSC**.

The paper is organized as follows. In Section 2, we obtain the dimensions of homogeneous sets. In Section 3, we show that Moran sets are homogeneous, and we also give many homogeneous sets which are not Ahlfors–David regular. In Section 4, we approximate the homogeneous sets by Moran sets. The bilipschitz embedding and quasi-Lipschitz equivalence of homogeneous sets are discussed in Sections 5 and 6 respectively.

2. DIMENSIONS OF HOMOGENEOUS FRACTALS

In this section, we will prove Proposition 1.

For a compact subset A in any metric space, let $P(A, r)$ denote the greatest number of disjoint r -balls with centers in A , and $N(A, r)$ the smallest number of r -balls needed to cover A . We have

$$N(A, 2r) \leq P(A, r) \leq N(A, r/2) \text{ for any } r > 0, \quad (2.1)$$

please refer to Section 5.3 of [13].

Proof of Proposition 1.

For any $r \leq |A|$, assume that $(\kappa_A)^n |A| < r \leq (\kappa_A)^{n-1} |A|$ ($n \geq 1$); then

$$\mu(B(x, (\kappa_A)^n |A|)) \leq \mu(B(x, r)) \leq \mu(B(x, (\kappa_A)^{n-1} |A|)).$$

Using (1.6), we have for any $x \in A$,

$$\frac{\mu(A)}{(\Delta_A)^n} = \frac{\mu(B(x, |A|))}{(\Delta_A)^n} \leq \mu(B(x, r)) \leq \frac{\mu(B(x, |A|))}{(\delta_A)^{n-1}} = \frac{\mu(A)}{(\delta_A)^{n-1}},$$

which implies

$$\frac{\log \delta_A}{-\log \kappa_A} \leq \liminf_{r \rightarrow 0} \alpha_A(r) \leq \limsup_{r \rightarrow 0} \alpha_A(r) \leq \frac{\log \Delta_A}{-\log \kappa_A}. \quad (2.2)$$

(1) Fix $x^* \in A$. For any $r > 0$, by (1.5) in Definition 1, we obtain

$$P(A, r) \cdot \lambda_A^{-1} \mu(B(x^*, r)) \leq \mu(A) \leq N(A, r) \cdot \lambda_A \mu(B(x^*, r)).$$

Then by (2.1), we have

$$\frac{\mu(A)}{\lambda_A \mu(B(x^*, r))} \leq N(A, r) \leq \frac{\lambda_A \mu(A)}{\mu(B(x^*, r/2))}.$$

It follows from (1.6) that μ is doubling, i.e., $\mu(B(x^*, r/2)) \geq C\mu(B(x^*, r))$ for some constant $C > 0$. Therefore,

$$f_A(r) = \frac{\log N(A, r)}{-\log r} \sim \alpha_A(x^*, r). \quad (2.3)$$

(2) Using definitions of dimensions (see e.g. [3] and [6]) and (2.3), we have

$$\dim_H A \leq \underline{\dim}_B A = \liminf_{r \rightarrow 0} \alpha_A(r), \quad \dim_P A \leq \overline{\dim}_B A = \limsup_{r \rightarrow 0} \alpha_A(r).$$

It suffices to show that

$$\dim_H A \geq \liminf_{r \rightarrow 0} \alpha_A(r) \quad \text{and} \quad \dim_P A \geq \limsup_{r \rightarrow 0} \alpha_A(r).$$

For any $0 < s < \liminf_{r \rightarrow 0} \alpha_A(r)$, there exists $r_0 \in (0, 1)$, such that for any $x \in A$ and $r \in (0, r_0]$,

$$\alpha_A(x, r) = \frac{\log \mu(B(x, r))}{\log r} > s.$$

Then for any subset $U \subset X$ with $A \cap U \neq \emptyset$ and $|U| \leq r_0$,

$$\mu(U) \leq \mu(B(x, |U|)) \leq |U|^s \text{ for any } x \in A \cap U.$$

In a standard way, we get $\dim_H A \geq \liminf_{r \rightarrow 0} \alpha_A(r)$.

By the Corollary 3.20(b) of [3], we have $\dim_P A \geq \limsup_{r \rightarrow 0} \alpha_A(r)$ directly. \square

3. MORAN SETS ARE HOMOGENEOUS

Given a Moran set $E \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$ in \mathbb{R}^d , for word $\sigma = i_1 \cdots i_k \in \Omega_k$ with length k , write $J_\sigma = J_{i_1 \cdots i_k}$, a basic element of order k . Without loss of generality, for the proof of Proposition 2, we may assume that $|J| = 1$. Let $r_k = c_1 \cdots c_k$ for all k , and let $c_1 \cdots c_{k-1} = n_1 \cdots n_{k-1} = 1$ for $k = 1$.

Let \mathcal{L} denote the Lebesgue measure on \mathbb{R}^d . Write $\text{int}(\cdot)$ the interior of set. Then

$$\mathcal{L}(\text{int}(J_\sigma)) = (r_k)^d \mathcal{L}(\text{int}(J)) \quad (3.1)$$

for $\sigma \in \Omega_k$ since J_σ is geometrically similar to J .

Notice that the union $\bigcup_{i_k=1}^{n_k} \text{int}(J_{i_1 \dots i_{k-1} i_k}) \subset \text{int}(J_{i_1 \dots i_{k-1}})$ is disjoint for any word $i_1 \dots i_{k-1} \in \Omega_{k-1}$, we have

$$\sum_{i_k=1}^{n_k} \mathcal{L}(\text{int}(J_{i_1 \dots i_{k-1} i_k})) \leq \mathcal{L}(\text{int}(J_{i_1 \dots i_{k-1}})).$$

Applying (3.1) to the above formula, we have

$$n_k c_k^d \leq 1. \quad (3.2)$$

Applying $c_* = \inf_k c_k > 0$ and $n_k \geq 2$ to (3.2), we have

$$c_* \leq c^* := \sup_k c_k \leq \frac{1}{\sqrt[d]{2}} \text{ and } 2 \leq n_k \leq c_*^{-d}. \quad (3.3)$$

3.1. Moran measure.

We are going to construct a Borel probability measure μ on \mathbb{R}^d with its support $\text{supp} \mu = E$ as in [2], which is usually called the Moran measure.

Let $\Omega^\infty = \prod_{k=1}^\infty \{1, \dots, n_k\}$ be a compact metrizable space. For $w = w_1 w_2 \dots \in \Omega^\infty$ and $k \geq 1$, let $w|_k = w_1 \dots w_k \in \Omega_k$. For $k \geq 1$ and $\sigma \in \Omega_k$, let $C_\sigma = \{w \in \Omega^\infty : w|_k = \sigma\}$, the cylinder set determined by σ . Then there is a unique Borel probability measure ν on Ω^∞ such that $\nu(C_\sigma) = (n_1 \dots n_k)^{-1}$ for all $k \geq 1$ and $\sigma \in \Omega_k$.

By (3.3), we notice that $r_k \rightarrow 0$ as $k \rightarrow \infty$, that is $\lim_{k \rightarrow \infty} |J_{w|_k}| = 0$. Thus there is a map $f: \Omega^\infty \rightarrow \mathbb{R}^d$ with $f(\Omega^\infty) = E$ defined by

$$\{f(w)\} = \bigcap_{k=1}^\infty J_{w|_k} \text{ for each } w \in \Omega^\infty;$$

and as $f(C_\sigma) \subset J_\sigma$ for each $\sigma \in \Omega^* = \bigcup_{k=0}^\infty \Omega_k$, the map f is continuous. Now there is a Borel probability measure μ on \mathbb{R}^d defined by $\mu(A) = \nu(f^{-1}(A))$ for all Borel set $A \subset \mathbb{R}^d$. Now

$$\mu(J_\sigma) = \nu(f^{-1}(J_\sigma)) \geq \nu(C_\sigma) = (n_1 \dots n_k)^{-1} \quad (3.4)$$

for all $k \geq 1$ and $\sigma \in \Omega_k$. From this it easily follows that the support of μ is E .

Next, we give an estimation of the Moran measure.

Lemma 3. *There is a constant $C_E > 1$ such that for any $x \in E$ and $r_k < r \leq r_{k-1}$,*

$$(n_1 \dots n_k)^{-1} \leq \mu(B(x, r)) \leq C_E (n_1 \dots n_{k-1})^{-1}.$$

Proof. Suppose J_σ is a basic element of order k containing x ; since $|J_\sigma| = r_k$, we have $J_\sigma \subset B(x, r)$. By (3.4) we have

$$\mu(B(x, r)) \geq \mu(J_\sigma) \geq (n_1 \dots n_k)^{-1}. \quad (3.5)$$

Let $\Lambda_{x,r} = \{\sigma' : \sigma' \in \Omega_{k-1} \text{ and } J_{\sigma'} \cap B(x, r) \neq \emptyset\}$. We will show that $\#\Lambda_{x,r} \leq C_E$ for some constant $C_E > 1$ independent of x and r .

Since $\text{int}(J_{\sigma'}) \cap \text{int}(J_{\sigma''}) = \emptyset$ for all distinct σ' and σ'' in $\Lambda_{x,r}$, and

$$\bigcup_{\sigma' \in \Lambda_{x,r}} \text{int}(J_{\sigma'}) \subset B(x, r + r_{k-1}) \subset B(x, 2r_{k-1}),$$

we have

$$\begin{aligned} (\#\Lambda_{x,r})\mathcal{L}(\text{int}(J))(r_{k-1})^d &= \sum_{\sigma' \in \Lambda_{x,r}} \mathcal{L}(\text{int}(J_{\sigma'})) \\ &= \mathcal{L}\left(\bigcup_{\sigma' \in \Lambda_{x,r}} \text{int}(J_{\sigma'})\right) \\ &\leq 2^d (r_{k-1})^d \mathcal{L}(B(0,1)), \end{aligned}$$

which implies $\#\Lambda_{x,r} \leq \frac{2^d \mathcal{L}(B(0,1))}{\mathcal{L}(\text{int}(J))} =: C_E$. Therefore,

$$\begin{aligned} \mu(B(x,r)) &= \nu(f^{-1}(B(x,r))) \\ &\leq \nu\{w \in \Omega^\infty : f(C_{w|_{k-1}}) \cap B(x,r) \neq \emptyset\} \\ &\leq \sum_{\sigma' \in \Lambda_{x,r}} \nu(C_{\sigma'}) \leq C_E (n_1 \cdots n_{k-1})^{-1}. \end{aligned} \tag{3.6}$$

Then this lemma follows from (3.5) and (3.6). \square

3.2. Proof of Proposition 2.

Using Lemma 3, we can prove that all Moran sets are homogeneous.

Proof of Proposition 2. Take $\lambda_E = C_E c_*^{-d}$. For any $x_1, x_2 \in E$, $r \in (0, |E|]$, if $r_k < r \leq r_{k-1}$ ($k \geq 1$), by Lemma 3, we have

$$\lambda_E^{-1} \leq \frac{1}{C_E n_k} \leq \frac{\mu(B(x_1, r))}{\mu(B(x_2, r))} \leq C_E n_k \leq \lambda_E.$$

Take $\kappa_A \in (0, 1)$ small enough such that

$$\delta_A := \frac{1}{C_E} \cdot 2^{\frac{\log \kappa_A}{\log c_*} - 2} > 1. \tag{3.7}$$

Assume that $r_k < r \leq r_{k-1}$ and $r_{k'} < \kappa_A r \leq r_{k'-1}$ with $k' \geq k$. Then $k' \geq k+1$ and

$$\frac{n_1 \cdots n_{k'-1}}{C_E n_1 \cdots n_k} \leq \frac{\mu(B(x, r))}{\mu(B(x, \kappa_A r))} \leq \frac{C_E n_1 \cdots n_{k'}}{n_1 \cdots n_{k-1}}, \tag{3.8}$$

where

$$\begin{aligned} \frac{C_E n_1 \cdots n_{k'}}{n_1 \cdots n_{k-1}} &\leq C_E (n_k \cdots n_{k'}) \leq C_E (c_*^{-d})^{k' - k + 1}, \\ \frac{n_1 \cdots n_{k'-1}}{C_E n_1 \cdots n_k} &\geq \frac{1}{C_E} (n_{k+1} \cdots n_{k'-1}) \geq \frac{1}{C_E} \cdot 2^{k' - k - 1}. \end{aligned} \tag{3.9}$$

Now, we have

$$(c_k \cdots c_{k'}) \frac{r_{k'}}{r_{k-1}} \leq \kappa_A \leq \frac{r_{k'-1}}{r_k} (= c_{k+1} \cdots c_{k'-1}),$$

which implies

$$(k' - k + 1) \log c_* \leq \log \kappa_A \leq (k' - k - 1) \log c^*,$$

i.e.,

$$\frac{\log \kappa_A}{\log c_*} - 1 \leq k' - k \leq \frac{\log \kappa_A}{\log c^*} + 1. \tag{3.10}$$

Let $\Delta_A = C_E(c_*^{-d})^{\frac{\log \kappa_A}{\log c_*}+2}$. Applying (3.7) and (3.10) to (3.8)-(3.9), we obtain

$$\delta_A \leq \frac{\mu(B(x, r))}{\mu(B(x, \kappa_A r))} \leq \Delta_A.$$

Lemma 3 and (3.3) shows that we can take

$$\alpha_E(r) = \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}$$

whenever $r_k < r \leq r_{k-1}$. \square

3.3. Moran sets which are not Ahlfors–David regular.

For Moran set E , it follows from Propositions 1-2 (also see [8] and [20]) that

$$\dim_H E = \liminf_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}, \quad \dim_P E = \limsup_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}. \quad (3.11)$$

Since $\dim_H F = \dim_P F$ for any Ahlfors–David regular set F , we have

Proposition 5. *If $\liminf_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k} < \limsup_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}$, then E is not Ahlfors–David regular for any $E \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$.*

Remark 9. *The above result shows that we can find lots of homogeneous sets which are not Ahlfors–David regular.*

The following example shows that a homogeneous set E with $\dim_H E = \dim_P E$ need not be Ahlfors–David regular.

Example 3. *Let $n_k \equiv 2$ and $c_k = \frac{k+1}{2(k+2)}$ for all $k \geq 1$. Denote $J = [0, 1]$, and let $J_1 = [0, c_1]$, $J_2 = [1 - c_1, 1]$. Inductively, if the interval $J_{i_1 \cdots i_k} = [c_{i_1 \cdots i_k}, d_{i_1 \cdots i_k}]$ have been defined, we define its subintervals $J_{i_1 \cdots i_k 1} = [c_{i_1 \cdots i_k}, c_{i_1 \cdots i_k} + c_k |J_{i_1 \cdots i_k}|]$ and $J_{i_1 \cdots i_k 2} = [d_{i_1 \cdots i_k} - c_k |J_{i_1 \cdots i_k}|, d_{i_1 \cdots i_k}]$. As above, we get a Moran set E . Using (3.11), we have $\dim_H E = \dim_P E = 1$. Notice that $\mathcal{H}^1(E) = \mathcal{L}(E) = 0$, where \mathcal{L} is the Lebesgue measure. If E is Ahlfors–David 1-regular, then $\mathcal{H}^1(E) > 0$, which is a contradiction. That means E is not Ahlfors–David regular.*

4. APPROXIMATING BY MORAN SETS

4.1. Bilipschitz image of homogeneous set.

Under bilipschitz mapping, the homogeneous property will be preserved.

Lemma 4. *Suppose $A \subset X$ is a homogeneous set. If A is bilipschitz equivalent to $B \subset Y$, then B is also homogeneous and $\chi(A, B) = 0$.*

Proof. Assume that f is the bilipschitz map from A onto B with bilipschitz constant $L \geq 1$, and μ is the corresponding measure supported on A . We define the image measure ν on B with $\nu(F) = \mu(f^{-1}(F))$ for any Borel subset $F \subset B$. It is clear that ν is a Borel measure supported on B .

Without loss of generality, we may assume that $A = X$ and $B = Y$, the whole metric spaces. For any $y \in B$, $0 < r \leq |B|$, we have

$$B(f^{-1}(y), r/L) \subset f^{-1}(B(y, r)) \subset B(f^{-1}(y), Lr); \quad (4.1)$$

then

$$\mu(B(f^{-1}(y), r/L)) \leq \nu(B(y, r)) \leq \mu(B(f^{-1}(y), Lr)). \quad (4.2)$$

Using Definition 1 and (4.2), for any $y_1, y_2 \in B$ and $r \leq |A|/L$, we have

$$\frac{\nu(B(y_1, r))}{\nu(B(y_2, r))} \leq \frac{\mu(B(f^{-1}(y_1), Lr))}{\mu(B(f^{-1}(y_2), r/L))} \leq \lambda_A \frac{\mu(B(f^{-1}(y_1), Lr))}{\mu(B(f^{-1}(y_1), r/L))} \leq \lambda_A (\Delta_A)^n,$$

where $(\kappa_A)^n \leq L^{-2} < (\kappa_A)^{n-1}$ for some integer $n \geq 0$. Then $n-1 < \frac{-2 \log L}{\log \kappa_A} \leq n$, and thus $n \leq 1 - \frac{2 \log L}{\log \kappa_A}$. Therefore, for all $y_1, y_2 \in B$ and $r \leq |A|/L$,

$$\frac{\nu(B(y_1, r))}{\nu(B(y_2, r))} \leq \lambda_A (\Delta_A)^n \leq \lambda := \lambda_A (\Delta_A)^{1 - \frac{2 \log L}{\log \kappa_A}},$$

which implies

$$\lambda^{-1} \leq \frac{\nu(B(y_1, r))}{\nu(B(y_2, r))} \leq \lambda \text{ for any } y_1, y_2 \in B, r \leq |A|/L.$$

Fix a point $y^* \in B$ and let $\lambda_B = \frac{\mu(A)}{\nu(B(y^*, |A|/L))} \cdot \lambda \geq \lambda$.

Given any $r \in [|A|/L, |B|]$, we have $\nu(B(y_1, r)) \leq \mu(A)$ and $\nu(B(y_2, r)) \geq \nu(B(y_2, |A|/L)) \geq \lambda^{-1} \nu(B(y^*, |A|/L))$, which implies

$$\lambda_B^{-1} \leq \frac{\nu(B(y_1, r))}{\nu(B(y_2, r))} \leq \lambda_B \text{ for any } y_1, y_2 \in B, r \leq |B|. \quad (4.3)$$

Let $\kappa_B = \kappa_A/L^2$. Using (4.2), for $r \leq |B| \leq |A| \cdot L$, for any $x \in A$ we have

$$\frac{\nu(B(f(x), r))}{\nu(B(f(x), \kappa_B r))} \geq \frac{\mu(B(x, r/L))}{\mu(B(x, \kappa_A r/L))} \geq \delta_A,$$

since $r/L \leq |A|$.

On the other hand, if $r \leq |A|/L$, then

$$\frac{\nu(B(f(x), r))}{\nu(B(f(x), \kappa_B r))} \leq \frac{\mu(B(x, rL))}{\mu(B(x, \kappa_A r/L^3))} \leq (\Delta_A)^m,$$

where $(\kappa_A)^m \leq \kappa_A L^{-4} < (\kappa_A)^{m-1}$ for some integer $m \geq 1$. Then $(\kappa_A)^{m-1} \leq L^{-4} < (\kappa_A)^{m-2}$, i.e., $m-2 \leq -\frac{4 \log L}{\log \kappa_A} \leq m-1$. Therefore,

$$\frac{\nu(B(f(x), r))}{\nu(B(f(x), \kappa_B r))} \leq (\Delta_A)^{2 - \frac{4 \log L}{\log \kappa_A}} \text{ for any } r \leq |A|/L.$$

Let $\Delta_B = \max((\Delta_A)^{2 - \frac{4 \log L}{\log \kappa_A}}, \frac{\lambda_B \mu(A)}{\nu(B(y^*, \kappa_B |A|/L))})$. Then we have

$$\frac{\nu(B(f(x), r))}{\nu(B(f(x), \kappa_B r))} \leq \Delta_B \text{ for all } r \leq |B|.$$

Therefore, for any $x \in A$ and $r \leq |B|$,

$$\delta_A \leq \frac{\nu(B(f(x), r))}{\nu(B(f(x), \kappa_B r))} \leq \Delta_B. \quad (4.4)$$

It follows from (4.3) and (4.4) that B is also homogeneous.

Using (4.1), we have

$$N(B, Lr) \leq N(A, r) \leq N(B, L^{-1}r). \quad (4.5)$$

It follows from (1.10), (4.5) and the fact that A is a doubling metric space that $\chi(A, B) = 0$. \square

Proof of Proposition 3.

In fact, suppose $\sup_{r < r_0} \alpha_A(r) < \infty$ for some r_0 small enough. We note that $\varphi(x) = e^x - 1$ is continuous at 0, then for fixed $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that if $\chi(A, B) = \limsup_{r \rightarrow 0} \left| \log \frac{\alpha_B(r)}{\alpha_A(r)} \right| < \delta$, then $\left| \frac{\alpha_B(r)}{\alpha_A(r)} - 1 \right| = |\varphi(\log \frac{\alpha_B(r)}{\alpha_A(r)})| < \varepsilon / (\sup_{r < r_0} \alpha_A(r))$ for all $r < r_1$ where $r_1 < r_0$ is a constant. Hence

$$|\alpha_B(r) - \alpha_A(r)| < \varepsilon \text{ for all } r < r_1,$$

and thus $|\overline{\lim}_{r \rightarrow 0} \alpha_B(r) - \overline{\lim}_{r \rightarrow 0} \alpha_A(r)|, |\underline{\lim}_{r \rightarrow 0} \alpha_B(r) - \underline{\lim}_{r \rightarrow 0} \alpha_A(r)| < \varepsilon$. It follows from (1) of Proposition 1 that $|\dim_P B - \dim_P A|, |\dim_H B - \dim_H A| < \varepsilon$. \square

4.2. Proof of the approximation theorem.

For homogeneous sets, we can approximate them by their subsets which are bilipschitz images of Moran sets in Euclidean spaces.

Proof of Theorem 1.

We can prove Theorem 1 in three steps:

- (1) For any $\varepsilon > 0$, choose η small enough and construct a subset $A(\eta)$ of A , such that $d_H(A(\eta), A) < \varepsilon$.
- (2) Corresponding to $A(\eta)$, construct a Moran set $E(\eta)$ in \mathbb{R}^d for some $d \in \mathbb{N}$. Show that the natural bijection between $A(\eta)$ and $E(\eta)$ is a bilipschitz map.
- (3) Verify that $\chi(E(\eta), A) = \chi(A(\eta), A) < \varepsilon$.

Without loss of generality, assume that A is homogeneous with $|A| = 1$. Let $\underline{\mu}(r) = \inf_{x \in A} \mu(B(x, r))$ and $\overline{\mu}(r) = \sup_{x \in A} \mu(B(x, r))$. Fix a point $x^* \in A$; then using (1.5) and (1.6), we have

$$\frac{\underline{\mu}(r/2)}{\overline{\mu}(2r')} \geq \frac{1}{(\lambda_A)^2} \frac{\mu(B(x^*, r/2))}{\mu(B(x^*, 2r'))} \geq \frac{1}{(\lambda_A)^2} (\delta_A)^n,$$

where $\frac{2r'}{r/2} \leq (\kappa_A)^n$ for some integer n . Taking n large enough, we have

$$\frac{\underline{\mu}(r/2)}{\overline{\mu}(2r')} \geq 2 \text{ if } \frac{r'}{r} \leq \eta_0 \quad (4.6)$$

for some constant η_0 . By Definition 1, there exists a constant $C_0 \in (0, 1)$ such that for any r, r' with $\frac{r'}{r} \leq \eta_0$,

$$C_0 \frac{\overline{\mu}(r)}{\overline{\mu}(r')} \leq \left\lfloor \frac{\underline{\mu}(r/2)}{\overline{\mu}(2r')} \right\rfloor \leq \frac{\overline{\mu}(r)}{\overline{\mu}(r')}, \quad (4.7)$$

where $\lfloor \cdot \rfloor$ denotes the integral part of number.

Step 1. Let $\varepsilon > 0$. Fix so large an integer n that

$$\eta_1 := (\kappa_A)^n < \min(\eta_0, \frac{1}{4}, \frac{\varepsilon}{3}) \text{ and } \left| \frac{\log C_0}{n \log \delta_A} \right| \leq \frac{\varepsilon}{2}. \quad (4.8)$$

Now choose $\eta > 0$ with $\eta \leq \eta_1$. Then $\overline{\mu}(\eta^k) \leq (\delta_A)^{-nk} \overline{\mu}(1)$ for each $k \geq 1$, and thus

$$\limsup_{k \rightarrow \infty} \left| \frac{(k-1) \log C_0}{\log \overline{\mu}(\eta^k)} \right| \leq \left| \frac{\log C_0}{n \log \delta_A} \right| \leq \frac{\varepsilon}{2}. \quad (4.9)$$

For all $k \geq 2$, let

$$n_k = \left\lfloor \frac{\underline{\mu}(\eta^{k-1}/2)}{\overline{\mu}(2\eta^k)} \right\rfloor \geq 2, \quad (4.10)$$

due to (4.6) as $\eta < \eta_0$.

We begin to construct the $A(\eta)$. In the first step of the construction, we get a maximal number $P_A = P(A, \eta)$ of disjoint η -balls $\{B(x_i, \eta)\}_{i=1}^{P_A}$ with centers in A . For a small enough η , let

$$n_1 = P_A \geq 2.$$

Given $\{n_k\}_k$ as above, let Ω^∞ denote the collection of all infinite sequences $i_1 \cdots i_k \cdots$ with $i_1 \cdots i_k \in \Omega_k$ for every $k \geq 1$.

For $k \geq 2$, inductively assume that for $k-1$, we have obtained a family of disjoint balls $\{B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1})\}_{i_1 \cdots i_{k-1} \in \Omega_{k-1}}$. We will find $\{B(x_{i_1 \cdots i_{k-1} i_k}, \eta^k)\}_{i_1 \cdots i_{k-1} i_k \in \Omega_k}$ satisfying for every $i_1 \cdots i_{k-1} \in \Omega_{k-1}$,

- $x_{i_1 \cdots i_{k-1} i_k} \in B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1}/2) \cap A$ for all $1 \leq i_k \leq n_k$;
- $B(x_{i_1 \cdots i_{k-1} i_k}, \eta^k) \cap B(x_{i_1 \cdots i_{k-1} j_k}, \eta^k) = \emptyset$ for all $i_k \neq j_k$.

In fact, fixing a sequence $i_1 \cdots i_{k-1} \in \Omega_{k-1}$, we take a maximal number $P_{i_1 \cdots i_{k-1}}$ of disjoint η^k -balls with centers in $B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1}/2) \cap A$. We will estimate $P_{i_1 \cdots i_{k-1}}$. At first, since $\eta < \frac{1}{4}$ by (4.8), for every η^k -ball $B(x, \eta^k)$ as above, we have

$$B(x, \eta^k) \subset B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1}/2 + \eta^k) \subset B(x_{i_1 \cdots i_{k-1}}, \frac{3}{4}\eta^{k-1}). \quad (4.11)$$

Since these $P_{i_1 \cdots i_{k-1}}$ disjoint η^k -balls are contained in $B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1})$, we have

$$P_{i_1 \cdots i_{k-1}} \leq \frac{\overline{\mu}(\eta^{k-1})}{\underline{\mu}(\eta^k)}. \quad (4.12)$$

On the other hand, by (2.1), $B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1}/2) \cap A$ can be covered by $P_{i_1 \cdots i_{k-1}}$ balls of radius $2\eta^k$, that means $\underline{\mu}(\eta^{k-1}/2) \leq P_{i_1 \cdots i_{k-1}} \cdot \overline{\mu}(2\eta^k)$, i.e.,

$$P_{i_1 \cdots i_{k-1}} \geq \frac{\underline{\mu}(\eta^{k-1}/2)}{\overline{\mu}(2\eta^k)} \geq n_k.$$

Hence we can take n_k disjoint η^k -balls with their centers in $B(x_{i_1 \cdots i_{k-1}}, \eta^{k-1}/2) \cap A$. Denote their centers by $\{x_{i_1 \cdots i_{k-1} i_k}\}_{i_k=1}^{n_k}$.

We define

$$A(\eta) = \bigcap_{k \geq 1} \bigcup_{i_1 \cdots i_k \in \Omega_k} B(x_{i_1 \cdots i_k}, \eta^k) \subset A. \quad (4.13)$$

For any $i_1 \cdots i_k \cdots \in \Omega^\infty$, let $x_{i_1 \cdots i_k \cdots} \in A(\eta)$ be such that

$$\{x_{i_1 \cdots i_k \cdots}\} = \bigcap_{k \geq 1} B(x_{i_1 \cdots i_k}, \eta^k). \quad (4.14)$$

Since in the first step of the construction of $A(\eta)$, we get the maximal number P_A of disjoint η -balls $\{B(x_i, \eta)\}_{i=1}^{P_A}$ with centers in A , it follows that A can be covered by P_A balls $\{B(x_i, 2\eta)\}_{i=1}^{P_A}$. Therefore

$$d_H(A(\eta), A) \leq d_H(\{x_i\}_{i=1}^{P_A}, A) + d_H(\{x_i\}_{i=1}^{P_A}, A(\eta)) \leq 2\eta + \eta < \varepsilon.$$

Step 2. For the η given, by Definition 1 from (4.12) it follows that $\{n_k\}_{k \geq 1}$ is bounded. Then taking d large enough, we can construct a Moran set $E(\eta)$ in \mathbb{R}^d such that $E(\eta) \in \mathcal{M}(J, \{n_k\}, \{c_k\})$ with $J = B(0, \frac{1}{2})$, n_k defined above, and $c_k \equiv \eta$ for all $k \geq 1$ such that there is a constant $c > 0$ for which

$$d(J_{i_1 \cdots i_{k-1} i_k}, J_{i_1 \cdots i_{k-1} j_k}) \geq c\eta^k \text{ for all } i_k \neq j_k.$$

For any $i_1 \cdots i_k \cdots \in \Omega^\infty$, let $y_{i_1 \cdots i_k \cdots} \in E(\eta)$ be such that

$$\{y_{i_1 \cdots i_k \cdots}\} = \bigcap_k J_{i_1 \cdots i_k}. \quad (4.15)$$

Naturally, we obtain a bijection f from $E(\eta)$ to $A(\eta)$ such that

$$f(y_{i_1 \dots i_k \dots}) = x_{i_1 \dots i_k \dots} \quad \forall i_1 \dots i_k \dots \in \Omega^\infty. \quad (4.16)$$

It suffices to show that f is bilipschitz. In fact, for distinct points $y' = y_{i_1 \dots i_{k-1} i_k \dots}$ and $y'' = y_{i_1 \dots i_{k-1} j_k \dots}$ with $i_k \neq j_k$ ($k \geq 1$), we have

$$c\eta^k \leq d(J_{i_1 \dots i_{k-1} i_k}, J_{i_1 \dots i_{k-1} j_k}) \leq |y' - y''| \leq |J_{i_1 \dots i_{k-1}}| = \eta^{k-1}. \quad (4.17)$$

On the other hand, $B(x_{i_1 \dots i_{k-1} i_k}, \eta^k)$ and $B(x_{i_1 \dots i_{k-1} j_k}, \eta^k)$ are disjoint and

$$x' = x_{i_1 \dots i_{k-1} i_k \dots} \in B(x_{i_1 \dots i_{k-1} i_k}, \frac{3}{4}\eta^k), \quad x'' = x_{i_1 \dots i_{k-1} j_k \dots} \in B(x_{i_1 \dots i_{k-1} j_k}, \frac{3}{4}\eta^k),$$

due to (4.11); therefore,

$$\frac{1}{4}\eta^k \leq d_X(x', x'') \leq |B(x_{i_1 \dots i_{k-1}}, \frac{3}{4}\eta^{k-1})| \leq \frac{3}{2}\eta^{k-1}. \quad (4.18)$$

It follows from (4.17) and (4.18) that f is bilipschitz.

Step 3. For the Moran set $E(\eta) \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$, $J = B(0, 1/2)$ with $|J| = 1$. Using Proposition 2, we can take

$$\alpha_{E(\eta)}(r) = \frac{\log n_1 \dots n_k}{-k \log \eta} \quad \text{for } \eta^k < r \leq \eta^{k-1},$$

where $C_0 \frac{\bar{\mu}(\eta^{k-1})}{\bar{\mu}(\eta^k)} \leq n_k \leq \frac{\bar{\mu}(\eta^{k-1})}{\bar{\mu}(\eta^k)}$ for $k \geq 2$ due to (4.7), which implies

$$\frac{\log \bar{\mu}(\eta^k)}{k \log \eta} - \left(\frac{\log n_1 \bar{\mu}(\eta)}{k \log \eta} + \frac{(k-1) \log C_0}{k \log \eta} \right) \leq \alpha_{E(\eta)}(r) \leq \frac{\log \bar{\mu}(\eta^k)}{k \log \eta} - \frac{\log n_1 \bar{\mu}(\eta)}{k \log \eta}.$$

Using (1.5) in Definition 1, for the homogeneous set A we have

$$\frac{\log \bar{\mu}(\eta^{k-1})}{k \log \eta} \leq \alpha_A(x_A, r) \leq \frac{\log \bar{\mu}(\eta^k)}{(k-1) \log \eta} \leq \frac{\log \bar{\mu}(\eta^k) - \log \lambda_A}{(k-1) \log \eta}. \quad (4.19)$$

It follows from (1.5) and (1.6) that $\bar{\mu}(\eta^{k-1}) \geq \bar{\mu}(\eta^k) \geq \varsigma \bar{\mu}(\eta^{k-1})$ for some constant $\varsigma > 0$, which implies

$$\frac{\log \bar{\mu}(\eta^{k-1})}{k \log \eta} - \frac{\log \bar{\mu}(\eta^k)}{k \log \eta}, \quad \frac{\log \bar{\mu}(\eta^k)}{(k-1) \log \eta} - \frac{\log \bar{\mu}(\eta^k)}{k \log \eta} = O\left(\frac{1}{k \log \eta}\right). \quad (4.20)$$

By (4.19) and (4.20), we can take a function $\alpha_A(r) \sim \alpha_A(x_A, r)$ defined by

$$\alpha_A(r) = \frac{\log \bar{\mu}(\eta^k)}{k \log \eta} \quad \text{for } \eta^k < r \leq \eta^{k-1}.$$

Using the inequality $|\log t| \leq \frac{3}{2}|t-1|$ for all $|t-1| \leq 1/3$, (4.9) and Lemma 4, we have, assuming $\varepsilon/2 \leq 1/3$ as we may, that

$$\begin{aligned} \chi(A(\eta), A) &= \chi(E(\eta), A) \\ &= \limsup_{r \rightarrow 0} \left| \log \frac{\alpha_{E(\eta)}(r)}{\alpha_A(r)} \right| \leq \frac{3}{2} \limsup_{r \rightarrow 0} \left| \frac{\alpha_{E(\eta)}(r)}{\alpha_A(r)} - 1 \right| \\ &\leq \frac{3}{2} \limsup_{k \rightarrow \infty} \left| \frac{\log n_1 \bar{\mu}(\eta)}{\log \bar{\mu}(\eta^k)} \right| + \frac{3}{2} \limsup_{k \rightarrow \infty} \left| \frac{(k-1) \log C_0}{\log \bar{\mu}(\eta^k)} \right| \\ &\leq 0 + \frac{3}{2} \cdot \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

In particular, if A is a homogeneous set in \mathbb{R}^d , since any two balls in \mathbb{R}^d are geometrically similar, the above construction shows that $A(\eta)$ is a Moran set. Take

$f = id$ and $F = A(\eta) \subset A$. Furthermore, using Proposition 3 we can approximate A by Moran sets simultaneously in three aspects: Hausdorff metric, Hausdorff dimension and packing dimension. \square

5. BILIPSCHITZ EMBEDDING OF HOMOGENEOUS SETS

5.1. Necessary condition of bilipschitz embedding.

As shown in [5], a self-similar set satisfying **SSC** can be bilipschitz embedded into any self-similar set with higher dimension.

However, for homogeneous fractals, we need the following new necessary condition (Lemma 1): if $A \hookrightarrow B$, then

$$\frac{\mu(B(x, r))}{\mu(B(x, r'))} \leq C \frac{\nu(B(y, r))}{\nu(B(y, r'))} \text{ for all } r' < r \leq \min(|A|, |B|), \quad (5.1)$$

where C is a constant.

Proof of Lemma 1.

Suppose that there is an injection $f: (A, d_A) \rightarrow (B, d_B)$ and a constant $L \geq 1$ such that for all $x_1, x_2 \in A$,

$$d_A(x_1, x_2)/L \leq d_B(f(x_1), f(x_2)) \leq L d_A(x_1, x_2).$$

Given positive quantities $\{\theta_\lambda\}_\lambda$ and $\{\vartheta_\lambda\}_\lambda$ with parameter λ , we say that they are comparable and denote $\theta_\lambda \asymp \vartheta_\lambda$, if there is a constant ρ independent of λ such that

$$\rho^{-1} \leq \frac{\theta_\lambda}{\vartheta_\lambda} \leq \rho.$$

For any subset \mathcal{C} of A , let $K_A(\mathcal{C}, r) = \max\{n : \text{there are } n \text{ distinct points } \{x_i\}_{i=1}^n \text{ of } \mathcal{C} \text{ such that } \min_{i \neq j} d_A(x_i, x_j) \geq r\}$. Therefore, for any $r' < r$,

$$K_A(B(x, r), r') \leq K_B(B(f(x), Lr), r'/L). \quad (5.2)$$

Using Definition 1, as in the proof of Proposition 1, we obtain that

$$K_A(B(x, r), r') \asymp P_A(B(x, r), r') \asymp N_A(B(x, r), r') \asymp \frac{\mu(B(x, r))}{\mu(B(x, r'))}, \quad (5.3)$$

where $P_A(\mathcal{C}, r) = \max\{n : \text{there are } n \text{ disjoint } r\text{-balls with centers in } \mathcal{C}\}$ and $N_A(\mathcal{C}, r) = \min\{n : \text{there are } n \text{ } r\text{-balls covering } \mathcal{C}\}$. Note that this result depends heavily on the fact that A is a doubling metric space.

In the same way, we obtain that for any $y \in B$,

$$K_B(B(f(x), Lr), r'/L) \asymp \frac{\nu(B(f(x), Lr))}{\nu(B(f(x), r'/L))} \asymp \frac{\nu(B(f(x), r))}{\nu(B(f(x), r'))} \asymp \frac{\nu(B(y, r))}{\nu(B(y, r'))}. \quad (5.4)$$

Thus (5.1) follows from (5.2)–(5.4).

By (5.1), we have

$$\sup_{r' < r_0} \sup_{r < r_0} \left| \frac{\log \mu(B(x, r)) - \log \mu(B(x, r'))}{\log \nu(B(y, r)) - \log \nu(B(y, r'))} \right| \leq 1 + \left| \frac{\log C}{\log \nu(B(y, r)) - \log \nu(B(y, r'))} \right|,$$

where C is an independent constant. Taking r_0 small enough, $\nu(B(y, r))/\nu(B(y, r'))$ is so large that $\sup_{r' < r_0} \sup_{r < r_0} \left| \frac{\log C}{\log \nu(B(y, r))/\nu(B(y, r'))} \right|$ is close to 0.

On the other hand, $|\alpha_A(r) \log r - \log \mu(B(x, r))|, |\alpha_B(r) \log r - \log \nu(B(y, r))| \leq C_1$ for some C_1 due to (1.8), and $\log \mu(B(x, r))/\mu(B(x, r'))$, $\log \nu(B(y, r))/\nu(B(y, r'))$ are arbitrarily large when r_0 is small enough. Thus

$$\sup_{r' < r_0 < r < r_0} \left| \frac{\alpha_A(r) \log r - \alpha_A(r') \log r'}{\alpha_B(r) \log r - \alpha_B(r') \log r'} \right| \leq 1 + \varepsilon(r_0), \quad (5.5)$$

with $\varepsilon(r_0) \downarrow 0$ as $r_0 \downarrow 0$. \square

Now we will construct Moran set B with number t such that for any Ahlfors–David regular set A satisfying $t < \dim_H A < \dim_H B$, the inequality (5.1) fails.

Proof of Proposition 4.

Let $t = \log 3 / \log 6$, $c_k \equiv 1/6$, $k_m \equiv m^3$ and $t_m = k_m + m$ for all m . We take

$$n_k = \begin{cases} 3 & \text{if } k \in [k_m + 1, t_m] \text{ for some } m, \\ 5 & \text{otherwise.} \end{cases}$$

Let $B \in \mathcal{M}([0, 1], \{n_k\}_k, \{c_k\}_k)$. Then it follows from Propositions 1 and 2 that B is homogeneous with

$$\dim_H B = \dim_P B = \lim_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k} = \frac{\log 5}{\log 6}.$$

Furthermore, assume that for every $i_1 \cdots i_{k-1} \in \Omega_{k-1}$, the subintervals

$$J_{i_1 \cdots i_{k-1} 1}, \dots, J_{i_1 \cdots i_{k-1} n_k}$$

are uniformly distributed in $J_{i_1 \cdots i_{k-1}}$ from left to right. Then $J_{i_1 \cdots i_{k-1}(\frac{n_k+1}{2})}$ and $J_{i_1 \cdots i_{k-1}}$ have the same middle point $y_{i_1 \cdots i_{k-1}}$.

For the Moran measure ν , we calculate that

$$\frac{\nu(B(y_{i_1 \cdots i_{k_m}}, (1/6)^{k_m}/2))}{\nu(B(y_{i_1 \cdots i_{k_m}}, (1/6)^{t_m}/2))} = 3^m. \quad (5.6)$$

Suppose A is Ahlfors–David s -regular with $s \in (\log 3 / \log 6, \log 5 / \log 6)$. Then for any $x \in A$,

$$\frac{\mu(B(x, (1/6)^{k_m}/2))}{\mu(B(x, (1/6)^{t_m}/2))} \geq \xi(6^m)^s \quad (5.7)$$

for some constant ξ .

If (5.1) were true, by (5.6) and (5.7) we would obtain that $s \leq \log 3 / \log 6$. It is a contradiction. \square

5.2. Proof of embedding theorem.

Before the proof of Theorem 2, we give a technical lemma as follows.

Suppose B is homogeneous with the Borel measure ν . Let

$$\overline{\nu}(r) = \sup_{x \in B} \nu(x, r) \text{ and } \underline{\nu}(r) = \inf_{x \in B} \nu(x, r).$$

Lemma 5. *Suppose that A and B are homogeneous sets. For any $\varepsilon > 0$ and $\eta > 0$ small enough, let $E(\eta) \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$ be the Moran set constructed in the proof of Theorem 1, which is bilipschitz equivalent to $A(\eta) \subset A$. If*

$$P(B, \eta) \geq n_1 \text{ and } \frac{\underline{\nu}(\eta^{k-1}/2)}{\overline{\nu}(2\eta^k)} \geq n_k \text{ for all } k \geq 2,$$

then $E(\eta) \hookrightarrow B$, and thus $A(\eta) \hookrightarrow B$.

We turn to the proof of Theorem 2.

Proof of the first part of Theorem 2.

Without loss of generality, we may assume that $|A| = |B| = 1$. Let η_1 be defined in (4.8). Using the above lemma, by (4.10) it suffices to show that if $\eta(\leq \eta_1)$ is small enough, then

$$n_1 = P_A \leq P_B = P(B, \eta) \quad (5.8)$$

and

$$\frac{\overline{\mu}(\eta^{k-1})}{\underline{\mu}(\eta^k)} \leq \frac{\underline{\nu}(\eta^{k-1}/2)}{\overline{\nu}(2\eta^k)} \text{ for all } k \geq 2. \quad (5.9)$$

To obtain (5.8), noticing that $P_A \asymp \frac{\mu(A)}{\mu(B(x^*, \eta))}$ and $P_B \asymp \frac{\nu(B)}{\nu(B(y^*, \eta))}$, we only need to check that

$$\limsup_{\eta \rightarrow 0} \left| \frac{\log \mu(B(x^*, \eta))}{\log \nu(B(y^*, \eta))} \right| = \limsup_{\eta \rightarrow 0} \frac{\alpha_A(\eta)}{\alpha_B(\eta)} < 1,$$

which follows from (1.15) by fixing r and letting $r' \rightarrow 0$.

To obtain (5.9), note that $\frac{\overline{\mu}(\eta^{k-1})}{\underline{\mu}(\eta^k)} \asymp \frac{\mu(B(x^*, \eta^{k-1}))}{\mu(B(x^*, \eta^k))}$ and $\frac{\underline{\nu}(\eta^{k-1}/2)}{\overline{\nu}(2\eta^k)} \asymp \frac{\nu(B(y^*, \eta^{k-1}))}{\nu(B(y^*, \eta^k))}$, we only need to find η_2 such that

$$\sup_{k \geq 1, \eta < \eta_2} \left| \frac{\log \mu(B(x^*, \eta^{k-1})) - \log \mu(B(x^*, \eta^k))}{\log \nu(B(y^*, \eta^{k-1})) - \log \nu(B(y^*, \eta^k))} \right| < 1. \quad (5.10)$$

In fact, as at the end of the proof of Lemma 1, we get (5.10) by using (1.15). \square

In order to prove the second part, we need the following easily proved key property [14] by Mattila and Saaranen on the decomposition of a uniformly disconnected set. The reader can refer to [18] for a proof.

Lemma 6. [14] *Suppose A is a uniformly disconnected compact subset of a metric space with constants $C > 1$ and $r^* > 0$ in (1.16). If E is a subset of A and $0 < r < r^*$ a number satisfying $d(E, A \setminus E) > Cr$, then there are sets $\{E_i\}_{i=1}^m$ and balls $\{B(x_i, r)\}_{i=1}^m$ satisfying*

- (1) $E = \bigcup_{i=1}^m E_i$;
- (2) $d(E_i, E_j) > r$ for all $i \neq j$;
- (3) $x_i \in E_i$ and $E \cap B(x_i, r) \subset E_i \subset B(x_i, Cr)$ for all i .

Suppose $\{r_k\}_{k \geq 1}$ is a sequence of positive numbers decreasing to zero with $r_1 < r^*$ and $r_k/r_{k+1} > C$ for all $k \geq 1$. We shall give a decomposition of the uniformly disconnected set A with respect to $\{r_k\}_{k \geq 1}$.

Set $\Lambda_0 = \{\emptyset\}$ with empty word \emptyset . Using Lemma 6 with $E = A$ and $r = r_1$ we get sets $\{A_{i_1}\}_{i_1=1}^{m_A}$ and balls $\{B(x_{i_1}, r_1)\}_{i_1=1}^{m_A}$ satisfying

- (1) $A = \bigcup_{i_1=1}^{m_A} A_{i_1}$;
- (2) $d(A_{i_1}, A_{j_1}) > r_1 (> Cr_2)$ for all $i_1 \neq j_1$;
- (3) $x_{i_1} \in A_{i_1}$ and $A \cap B(x_{i_1}, r_1) \subset A_{i_1} \subset B(x_{i_1}, Cr_1)$ for all i_1 .

Set $m_\emptyset = m_A$ and $\Lambda_1 = \{1, 2, \dots, m_\emptyset\}$.

For $k \geq 2$, assume that for $k-1$ we have got the sets $\{A_{i_1 \dots i_{k-1}}\}_{i_1 \dots i_{k-1} \in \Lambda_{k-1}}$ and balls $\{B(x_{i_1 \dots i_{k-1}}, r_{k-1})\}_{i_1 \dots i_{k-1} \in \Lambda_{k-1}}$. By induction, we can do the same work to every $A_{i_1 \dots i_{k-1}}$ with $r = r_k$. Let $\Lambda_k = \{i_1 \dots i_k : i_j \in \mathbb{N} \cap [1, m_{i_1 \dots i_{j-1}}] \text{ for all } 1 \leq j \leq k\}$.

Since $r_k/r_{k+1} > C$, using Lemma 6 again and again, we get the decomposition of A . There exist sets $A_{i_1 \dots i_k}$ and points $x_{i_1 \dots i_k}$ such that for all $k \geq 1$,

$$\begin{aligned} A &= \bigcup_{i_1 \dots i_k \in \Lambda_k} A_{i_1 \dots i_k}, \\ d(A_{i_1 \dots i_{k-1} i_k}, A_{i_1 \dots i_{k-1} j_k}) &> r_k (> Cr_{k+1}) \text{ if } i_k \neq j_k, \\ A_{i_1 \dots i_k i_{k+1}} &\subset A_{i_1 \dots i_k}, \\ x_{i_1 \dots i_k} &\in A_{i_1 \dots i_k}, \\ A \cap B(x_{i_1 \dots i_k}, r_k) &\subset A_{i_1 \dots i_k} \subset B(x_{i_1 \dots i_k}, Cr_k). \end{aligned}$$

We denote $\Lambda = \bigcup_{k \geq 0} \Lambda_k$.

Proof of the second part of Theorem 2.

For any $\eta \in (0, \min(1/C, r^*))$, we get the decomposition of A with respect to $\{\eta^k\}_{k \geq 1}$. Note that $m_A \leq \frac{\mu(A)}{\underline{\mu}(\eta)}$ and $m_{i_1 \dots i_k} \leq \frac{\overline{\mu}(C\eta^k)}{\underline{\mu}(\eta^{k+1})}$ for all $i_1 \dots i_k \in \Lambda \setminus \Lambda_0$. Hence, as in the first part of the proof, there exists an $\eta_3 \in (0, 1/C)$, such that if η satisfies $\eta < \min\{\eta_1, \eta_3\}$, then

$$m_A \leq P(B, \eta) \text{ and } \frac{\overline{\mu}(C\eta^{k-1})}{\underline{\mu}(\eta^k)} \leq \frac{\underline{\nu}(\eta^{k-1}/2)}{\overline{\nu}(2\eta^k)} \text{ for all } k \geq 2.$$

Corresponding to the decomposition of A , we get a collection $\{B(y_{i_1 \dots i_k}, \eta^k)\}_{i_1 \dots i_k \in \Lambda \setminus \Lambda_0}$ of balls in B as in Step 1 in the proof of Theorem 1, satisfying

- (1) For every $i_1 \in \Lambda_1$, $y_{i_1} \in B$, and $B(y_{i_1}, \eta) \cap B(y_{j_1}, \eta) = \emptyset$ for all $i_1 \neq j_1$;
- (2) When $k \geq 2$, for every $i_1 \dots i_{k-1} \in \Lambda_{k-1}$,
 - $y_{i_1 \dots i_{k-1} i_k} \in B(y_{i_1 \dots i_{k-1}}, \eta^{k-1}/2) \cap B$ for all $1 \leq i_k \leq m_{i_1 \dots i_{k-1}}$;
 - $B(y_{i_1 \dots i_{k-1} i_k}, \eta^k) \cap B(y_{i_1 \dots i_{k-1} j_k}, \eta^k) = \emptyset$ for all $i_k \neq j_k$.

We define

$$B(\eta) = \bigcap_{k \geq 1} \bigcup_{i_1 \dots i_k \in \Lambda_k} B(y_{i_1 \dots i_k}, \eta^k).$$

Noting that

$$A = \bigcap_{k \geq 1} \bigcup_{i_1 \dots i_k \in \Lambda_k} A_{i_1 \dots i_k},$$

we can check as in Step 2 in the proof of Theorem 1 that the natural bijection between A and $B(\eta)$ is bilipschitz. \square

5.3. Uniform disconnectedness.

In the following proof, we use the idea of [14] by Mattila and Saaranen.

Proof of Lemma 2.

By (1.8), we may assume that there exists $r_0 \in (0, 1)$ such that

$$\sup_{r' < r_0 r'' < r'' < r_0} \left| \frac{\log \overline{\mu}(r'') - \log \underline{\mu}(r')}{\log r'' - \log r'} \right| \leq 1 - \gamma \text{ with } \gamma > 0. \quad (5.11)$$

Take an integer l large enough such that

$$\frac{\log l - \log 3}{\log(l+2)} > 1 - \gamma \text{ and } \frac{1}{l+2} < r_0. \quad (5.12)$$

For any $r < r_0/(l+2)$ and $x \in A$, let

$$B_0 = B(x, r), \quad B_i = B(x, (i+1)r) \setminus B(x, ir) \quad (1 \leq i \leq l+1).$$

As in [14], we only need to verify

Claim 1. *There must be an $i_0 \in \{1, \dots, l\}$ such that $A \cap B_{i_0} = \emptyset$.*

Otherwise, there exists $x_i \in A \cap B_i$ whenever $1 \leq i \leq l$. Then

$$\begin{aligned} l\mu(r) &\leq \sum_{i=1}^l \mu(B(x_i, r)) \leq \sum_{i=1}^l \mu(B_{i-1} \cup B_i \cup B_{i+1}) \\ &\leq 3\mu(B(x, (l+2)r)) \leq 3\bar{\mu}((l+2)r). \end{aligned}$$

Therefore,

$$1 - \gamma < \frac{\log l - \log 3}{\log(l+2)} \leq \left| \frac{\log \bar{\mu}((l+2)r) - \log \mu(r)}{\log(l+2)r - \log r} \right| \leq 1 - \gamma.$$

This is a contradiction. Then the claim is proved, and thus (1.16) holds with $C = l$. That means A is uniformly disconnected. \square

We will construct a Moran set E such that $\dim_H E = \dim_P E < 1$ but E is not uniformly disconnected.

Example 4. *Let $k_m = m^3$ and $t_m = k_m + m$ for all m . We take*

$$(n_k, c_k) = \begin{cases} (3, 1/3 - 1/(6m)) & \text{if } k \in [k_m + 1, t_m] \text{ for some } m, \\ (3, 1/6) & \text{otherwise.} \end{cases}$$

Let $E \in \mathcal{M}([0, 1], \{n_k\}_k, \{c_k\}_k)$. Then it follows from Propositions 2 and 1 that E is homogeneous with

$$\dim_H E = \dim_P E = \lim_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k} = \frac{\log 3}{\log 6} < 1.$$

Assume that for every word $i_1 \cdots i_{k-1}$, the subintervals $J_{i_1 \cdots i_{k-1}1}, \dots, J_{i_1 \cdots i_{k-1}n_k}$ are uniformly distributed in $J_{i_1 \cdots i_{k-1}}$ from left to right. If we consider the middle point $1/2$ and the largest gap in the interval $J_{i_1 \cdots i_{k_m}}$ with $1/2 \in J_{i_1 \cdots i_{k_m}}$, then, since $1 - 3c_{(k_m+1)} \rightarrow 0$ as $m \rightarrow \infty$, we clearly see that we can not find a uniform disconnectedness constant $C > 1$.

6. QUASI-LIPSCHITZ EQUIVALENCE OF HOMOGENEOUS SETS

In this section, we will prove Theorem 3. Without loss of generality, we always assume that $A = X$ and $B = Y$. We say that when $r, r' \rightarrow 0$,

$$g(r, r') \rightarrow a \Leftrightarrow \bar{g}(r, r') \rightarrow b,$$

if for any $\epsilon > 0$ there exists an $\eta > 0$ such that $|\bar{g}(r, r') - b| < \epsilon$ whenever $\max(|g(r, r') - a|, |r|, |r'|) < \eta$ and such that $|g(r, r') - a| < \epsilon$ whenever $\max(|\bar{g}(r, r') - b|, |r|, |r'|) < \eta$.

Lemma 7. *For any $x \in A$, when $r, r' \rightarrow 0$,*

$$\frac{\log r'}{\log r} \rightarrow 1 \Leftrightarrow \frac{\log \mu(B(x, r'))}{\log \mu(B(x, r))} \rightarrow 1.$$

Proof. Suppose that $r \in ((\kappa_A)^k, (\kappa_A)^{k-1}]$ and $r' \in ((\kappa_A)^{k'}, (\kappa_A)^{k'-1}]$. Then

$$\frac{\log r}{k \log \kappa_A} \rightarrow 1, \frac{\log r'}{k' \log \kappa_A} \rightarrow 1 \text{ as } r, r' \rightarrow 0.$$

On the other hand, $\mu(B(x, (\kappa_A)^k)) \leq \mu(B(x, r)) \leq \mu(B(x, (\kappa_A)^{k-1}))$ and

$$\mu(B(x, (\kappa_A)^k)) \geq \Delta_A^{-1} \mu(B(x, (\kappa_A)^{k-1}))$$

due to (1.6) in Definition 1. Thus

$$\frac{\log \mu(B(x, r))}{\log \mu(B(x, (\kappa_A)^k))} \rightarrow 1, \frac{\log \mu(B(x, r'))}{\log \mu(B(x, (\kappa_A)^{k'}))} \rightarrow 1 \text{ as } r, r' \rightarrow 0.$$

It suffices to verify that when $k, k' \rightarrow \infty$,

$$\frac{k'}{k} \rightarrow 1 \Leftrightarrow \frac{\log \mu(B(x, (\kappa_A)^{k'}))}{\log \mu(B(x, (\kappa_A)^k))} \rightarrow 1.$$

For $k > k'$, using (1.6), we have $(\delta_A)^{k-k'} \leq \frac{\mu(B(x, (\kappa_A)^{k'}))}{\mu(B(x, (\kappa_A)^k))} \leq (\Delta_A)^{k-k'}$, i.e.,

$$(k - k') \log \delta_A \leq \log \frac{\mu(B(x, (\kappa_A)^{k'}))}{\mu(B(x, (\kappa_A)^k))} \leq (k - k') \log \Delta_A. \quad (6.1)$$

Using (6.1) and $0 < \lim_{k \rightarrow \infty} \alpha_A(x, (\kappa_A)^k) \leq \overline{\lim}_{k \rightarrow \infty} \alpha_A(x, (\kappa_A)^k) < \infty$, we have

$$\begin{aligned} \frac{\log \mu(B(x, (\kappa_A)^{k'}))}{\log \mu(B(x, (\kappa_A)^k))} \rightarrow 1 &\Leftrightarrow \frac{\log \frac{\mu(B(x, (\kappa_A)^{k'}))}{\mu(B(x, (\kappa_A)^k))}}{\log \mu(B(x, (\kappa_A)^k))} \rightarrow 0 \\ &\Leftrightarrow \left(\frac{k - k'}{k} \right) \frac{\log(\kappa_A)^k}{\log \mu(B(x, (\kappa_A)^k))} \rightarrow 0 \\ &\Leftrightarrow \left(\frac{k - k'}{k} \right) \frac{1}{\alpha_A(x, (\kappa_A)^k)} \rightarrow 0 \\ &\Leftrightarrow \frac{k'}{k} \rightarrow 1. \end{aligned}$$

□

6.1. Proof of equivalence theorem: necessity.

By the definition of quasi-Lipschitz equivalence, we can find a bijection $f : A \rightarrow B$ and a non-decreasing function $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{r \rightarrow 0} \beta(r) = 0$ such that for every pair of distinct points $x_1, x_2 \in A$,

$$1 - \beta(d_A(x_1, x_2)) \leq \frac{\log d_B(f(x_1), f(x_2))}{\log d_A(x_1, x_2)} \leq 1 + \beta(d_A(x_1, x_2)) \quad (6.2)$$

and

$$1 - \beta(d_B(f(x_1), f(x_2))) \leq \frac{\log d_A(x_1, x_2)}{\log d_B(f(x_1), f(x_2))} \leq 1 + \beta(d_B(f(x_1), f(x_2))). \quad (6.3)$$

For any $x \in A$ and $r > 0$ small enough, we conclude that

$$B(f(x), r) \subset f(B(x, r^{1-\beta(r)})) \text{ and } f(B(x, r)) \subset B(f(x), r^{1-\beta(r)}). \quad (6.4)$$

In fact, we assume that r and $\beta(r)$ are small enough. Firstly, we verify that $B(f(x), r) \subset f(B(x, r^{1-\beta(r)}))$. For any $f(x') \in B(f(x), r)$ with $f(x') \neq f(x)$, we have $0 < d_B(f(x), f(x')) \leq r$. By (6.3), we have

$$\frac{\log d_A(x, x')}{\log d_B(f(x), f(x'))} \geq 1 - \beta(d_B(f(x), f(x'))) \geq 1 - \beta(r),$$

since the function β is non-decreasing,

$$d_A(x, x') \leq (d_B(f(x), f(x')))^{1-\beta(r)} \leq r^{1-\beta(r)}.$$

Then $x' \in B(x, r^{1-\beta(r)})$, and thus $B(f(x), r) \subset f(B(x, r^{1-\beta(r)}))$. In the same way, using (6.2), we have $f(B(x, r)) \subset B(f(x), r^{1-\beta(r)})$.

Using (6.4), we have

$$N(B, r^{1-\beta(r)}) \leq N(A, r) \text{ and } N(A, r^{1-\beta(r)}) \leq N(B, r).$$

Since $\beta(r) \downarrow 0$ when $r \downarrow 0$, using (2) of Proposition 1, we have

$$\chi(A, B) = 0.$$

6.2. Proof of equivalence theorem: sufficiency.

Without loss of generality we may assume that $|A| = |B| = 1$. Let $\Sigma = \{0, 1\}^{\mathbb{N}} = \{w_1 w_2 \cdots : w_i \in \{0, 1\} \text{ for all } i \geq 1\}$ be a symbolic system equipped with the metric $D(x, y) = 2^{-\min\{i \in \mathbb{N} : w_i \neq \omega_i\}}$ for distinct points $x = w_1 w_2 \cdots$, $y = \omega_1 \omega_2 \cdots$. Given two words $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ and an infinite word $w = w_1 w_2 \cdots$, we write

$$u * v = u_1 \cdots u_m v_1 \cdots v_n \quad \text{and} \quad u * w = u_1 \cdots u_m w_1 w_2 \cdots.$$

The set $\{u * w : w \in \Sigma\}$ is called the cylinder determined by u , and the length m of u is called the length of this cylinder.

Choose any $\eta \in (0, \min(1/C, r^*))$, where C is the uniform disconnectedness constant of A . Then we can get a decomposition of A with respect to $\{\eta^{k^2}\}_{k \geq 1}$ (see the discussion after Lemma 6). Corresponding to the decomposition, we will give a decomposition of Σ and construct a quasi-Lipschitz bijection from Σ onto A . With the same work to B , we can prove that the resulting bijection between A and B is quasi-Lipschitz.

Now, for all $k \geq 1$ we have

$$\frac{\underline{\mu}(\eta^{(k-1)^2})}{\overline{\mu}(C\eta^{k^2})} \leq m_{i_1 \cdots i_{k-1}} \leq \frac{\overline{\mu}(C\eta^{(k-1)^2})}{\underline{\mu}(\eta^{k^2})}. \quad (6.5)$$

By choosing η small enough we may assume that $m_{i_1 \cdots i_{k-1}} > 2$ for each $k \geq 1$. Assume that $p_{i_1 \cdots i_{k-1}} \geq 1$ is the integer satisfying

$$2^{p_{i_1 \cdots i_{k-1}}} < m_{i_1 \cdots i_{k-1}} \leq 2^{1+p_{i_1 \cdots i_{k-1}}}. \quad (6.6)$$

Step 1. According to the decomposition of A , we give a decomposition of Σ .

Set $\Sigma_\emptyset = \Sigma$ and $l_\emptyset = 0$. Denote all the words in $\{0, 1\}^{p_\emptyset}$ by $\pi_1, \dots, \pi_{2^{p_\emptyset}}$. Then the words

$$\pi_1 * 0, \pi_1 * 1, \dots, \pi_m * 0, \pi_m * 1, \pi_{m+1}, \dots, \pi_{2^{p_\emptyset}}$$

give m_\emptyset cylinders whose union is Σ , where $m = m_\emptyset - 2^{p_\emptyset}$.

We denote these cylinders by $\{\Sigma_{i_1}\}_{i_1 \in \Lambda_1}$ with lengths $\{l_{i_1}\}_{i_1 \in \Lambda_1}$. It is clear that

- (1) $l_{i_1} = p_\emptyset$ or $1 + p_\emptyset$ for all $1 \leq i_1 \leq m_\emptyset$;
- (2) $D(\Sigma_{i_1}, \Sigma_{j_1}) \geq 2^{-(1+p_\emptyset)}$ for all $i_1 \neq j_1$.

For $k \geq 2$, as usual, inductively assume that for $k-1$, we have got the cylinders $\{\Sigma_{i_1 \cdots i_{k-1}}\}_{i_1 \cdots i_{k-1} \in \Lambda_{k-1}}$ with lengths $\{l_{i_1 \cdots i_{k-1}}\}_{i_1 \cdots i_{k-1} \in \Lambda_{k-1}}$. With the same work to every $\Sigma_{i_1 \cdots i_{k-1}}$, we can find $m_{i_1 \cdots i_{k-1}}$ cylinders $\Sigma_{i_1 \cdots i_{k-1} i_k}$ with lengths $l_{i_1 \cdots i_{k-1} i_k}$ satisfying

- (1) $\Sigma_{i_1 \cdots i_{k-1}} = \bigcup_{i_k=1}^{m_{i_1 \cdots i_{k-1}}} \Sigma_{i_1 \cdots i_{k-1} i_k}$;
- (2) $D(\Sigma_{i_1 \cdots i_{k-1} i_k}, \Sigma_{i_1 \cdots i_{k-1} j_k}) \geq 2^{-l_{i_1 \cdots i_{k-1}} - (1+p_{i_1 \cdots i_{k-1}})}$ if $i_k \neq j_k$;
- (3) $l_{i_1 \cdots i_{k-1} i_k} - l_{i_1 \cdots i_{k-1}} = p_{i_1 \cdots i_{k-1}}$ or $1 + p_{i_1 \cdots i_{k-1}}$.

Then we get the decomposition of Σ . There exist cylinders $\Sigma_{i_1 \dots i_k}$ of lengths $l_{i_1 \dots i_k}$ such that for all $k \geq 1$,

$$\begin{aligned} \Sigma &= \bigcup_{i_1 \dots i_k \in \Lambda_k} \Sigma_{i_1 \dots i_k}, \\ D(\Sigma_{i_1 \dots i_{k-1} i_k}, \Sigma_{i_1 \dots i_{k-1} j_k}) &\geq 2^{-l_{i_1 \dots i_{k-1} i_k} - (1 + p_{i_1 \dots i_{k-1}})} \text{ if } i_k \neq j_k, \\ \Sigma_{i_1 \dots i_k i_{k+1}} &\subset \Sigma_{i_1 \dots i_k}, \\ l_{i_1 \dots i_{k-1} i_k} - l_{i_1 \dots i_{k-1}} &= p_{i_1 \dots i_{k-1}} \text{ or } 1 + p_{i_1 \dots i_{k-1}}, \\ l_{i_1 \dots i_k} &\geq k. \end{aligned}$$

Step 2. To verify the existence of the desired bijection between A and B , we construct a bijection f from Σ onto A .

Let Λ^∞ be the collection of the infinite words $i_1 \dots i_k \dots$ with $i_1 \dots i_k \in \Lambda_k$ for all k . For any $i_1 \dots i_k \dots \in \Lambda^\infty$, let $x_{i_1 \dots i_k \dots} \in A$ and $w_{i_1 \dots i_k \dots} \in \Sigma$ be such that

$$\{x_{i_1 \dots i_k \dots}\} = \bigcap_{k \geq 1} A_{i_1 \dots i_k} \quad \text{and} \quad \{w_{i_1 \dots i_k \dots}\} = \bigcap_{k \geq 1} \Sigma_{i_1 \dots i_k};$$

note that $|\Sigma_{i_1 \dots i_k}| \leq 2^{-k-1} \rightarrow 0$ as $k \rightarrow \infty$. In a natural way, we obtain a bijection f from Σ onto A , such that for any $i_1 \dots i_k \dots \in \Lambda^\infty$,

$$f(w_{i_1 \dots i_k \dots}) = x_{i_1 \dots i_k \dots}.$$

In the next step, we will prove that for any distinct points $z_1, z_2 \in \Sigma$,

$$\frac{\alpha_A(d_A(f(z_1), f(z_2))) \log d_A(f(z_1), f(z_2))}{\log D(z_1, z_2)} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0. \quad (6.7)$$

Then we can also construct a bijection g from Σ onto B in the same way, with

$$\frac{\alpha_B(d_B(g(z_1), g(z_2))) \log d_B(g(z_1), g(z_2))}{\log D(z_1, z_2)} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0. \quad (6.8)$$

Now, we get a bijection $g \circ f^{-1}$ from A onto B . Using (6.7)–(6.8), Lemma 7 and the assumption $\chi(A, B) = 0$, we have

$$\frac{\log d_B(g(z_1), g(z_2))}{\log d_A(f(z_1), f(z_2))} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0. \quad (6.9)$$

Hence A and B are quasi-Lipschitz equivalent.

We will give the details of (6.9) as follows:

According to the decompositions of A and Σ , we know that the bijection f is continuous and thus is uniformly continuous. That is

$$d_A(f(z_1), f(z_2)) \rightarrow 0 \text{ uniformly as } D(z_1, z_2) \rightarrow 0.$$

For the same reason,

$$d_B(g(z_1), g(z_2)) \rightarrow 0 \text{ uniformly as } D(z_1, z_2) \rightarrow 0.$$

Firstly, by (6.4)–(6.5), we have

$$\frac{\alpha_A(d_A(f(z_1), f(z_2))) \log d_A(f(z_1), f(z_2))}{\alpha_B(d_B(g(z_1), g(z_2))) \log d_B(g(z_1), g(z_2))} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0.$$

By the definition of χ , we have $\chi(A, B) = 0$ if and only if $\lim_{r \rightarrow 0} \frac{\alpha_A(r)}{\alpha_B(r)} = 1$; then we get that

$$\frac{\alpha_A(d_B(g(z_1), g(z_2)))}{\alpha_B(d_B(g(z_1), g(z_2)))} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0.$$

By the above two formulas, we obtain that

$$\frac{\alpha_A(d_A(f(z_1), f(z_2))) \log d_A(f(z_1), f(z_2))}{\alpha_A(d_B(g(z_1), g(z_2))) \log d_B(g(z_1), g(z_2))} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0,$$

that is

$$\frac{\log \mu(B(x, d_A(f(z_1), f(z_2))))}{\log \mu(B(x, d_B(g(z_1), g(z_2))))} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0, \quad (6.10)$$

for some fixed $x \in A$. Finally, by (6.10) and Lemma 7, we get (6.9).

Step 3. We need to check (6.7).

For any given different points $z_1, z_2 \in \Sigma$, suppose $i_1 \cdots i_{k-1}$ ($k \geq 1$) is the longest word such that $A_{i_1 \cdots i_{k-1}}$ contains both $f(z_1)$ and $f(z_2)$. Then $f(z_1) \in A_{i_1 \cdots i_{k-1} i_k}$, $f(z_2) \in A_{i_1 \cdots i_{k-1} j_k}$ with $i_k \neq j_k$. By (6.5) and (6.6),

$$\begin{aligned} D(z_1, z_2) &\geq 2^{-l_{i_1 \cdots i_{k-1}} - (1 + p_{i_1 \cdots i_{k-1}})} \\ &= 2^{-(1 + p_{i_1 \cdots i_{k-1}}) - (l_{i_1 \cdots i_{k-1}} - l_{i_1 \cdots i_{k-2}}) - \cdots - (l_{i_1} - l_\emptyset) - l_\emptyset} \\ &\geq \prod_{i=1}^k \frac{\underline{\mu}(\eta^{i^2})}{2\overline{\mu}(C\eta^{(i-1)^2})}. \end{aligned} \quad (6.11)$$

In the same way,

$$D(z_1, z_2) \leq 2^{-l_{i_1 \cdots i_{k-1}}} \leq \prod_{i=1}^{k-1} \frac{2\overline{\mu}(C\eta^{i^2})}{\underline{\mu}(\eta^{(i-1)^2})}. \quad (6.12)$$

Then

$$(I) \leq \log D(z_1, z_2) \leq (II), \quad (6.13)$$

where

$$(I) = \log \underline{\mu}(\eta^{k^2}) - k \log 2 - \log \overline{\mu}(C) + \sum_{i=1}^{k-1} \left(\log \underline{\mu}(\eta^{i^2}) - \log \overline{\mu}(C\eta^{i^2}) \right) \quad (6.14)$$

and

$$(II) = \log \overline{\mu}(C\eta^{k^2}) + (k-1) \log 2 + \sum_{i=1}^{k-2} \left(\log \overline{\mu}(C\eta^{i^2}) - \log \underline{\mu}(\eta^{i^2}) \right). \quad (6.15)$$

By Definition 1, here $\log \underline{\mu}(\eta^{k^2}) \geq -ak^2 + b$, $\log \overline{\mu}(C\eta^{k^2}) \leq -a'k^2 + b'$ and

$$0 \leq \log \overline{\mu}(C\eta^{i^2}) - \log \underline{\mu}(\eta^{i^2}) \leq c \quad \text{if } i \geq 1$$

with some constants $a, a' > 0$, $b, b' \in \mathbb{R}$ and $c > 0$. Therefore, $k \rightarrow \infty$, uniformly as $D(z_1, z_2) \rightarrow 0$.

Notice that

$$\eta^{k^2} \leq d_A(f(z_1), f(z_2)) \leq |A_{i_1 \cdots i_{k-1}}| \leq |B(x_{i_1 \cdots i_{k-1}}, C\eta^{(k-1)^2})| \leq 2C\eta^{(k-1)^2};$$

then

$$\frac{\log d_A(f(z_1), f(z_2))}{k^2 \log \eta} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0. \quad (6.16)$$

By (6.13)–(6.15) and the estimates related to (6.14)–(6.15), we have

$$\frac{\log D(z_1, z_2)}{\alpha_A(\eta^{k^2}) \cdot k^2 \log \eta} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0. \quad (6.17)$$

On the other hand, by (6.16) and Lemma 7 we have

$$\frac{\alpha_A(d_A(f(z_1), f(z_2)))}{\alpha_A(\eta^{k^2})} \rightarrow 1 \text{ uniformly as } D(z_1, z_2) \rightarrow 0. \quad (6.18)$$

Now (6.16)–(6.18) imply (6.7).

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SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, 430074, WUHAN, P. R. CHINA

E-mail address: lvfan1123@163.com

DEPARTMENT OF MATHEMATICS, GUANGDONG POLYTECHNIC NORMAL UNIVERSITY, GUANGZHOU, 510665, P. R. CHINA

E-mail address: loumanli@126.com

DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY, 100084, BEIJING, P. R. CHINA

E-mail address: wenzy@tsinghua.edu.cn

INSTITUTE OF MATHEMATICS, ZHEJIANG WANLI UNIVERSITY, 315100, NINGBO, P. R. CHINA

E-mail address: xilifengningbo@yahoo.com